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COMPACTNESS, INTERPOLATION AND FRIEDMAN'S THIRD PROBLEM

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Let Robinson's consistency theorem hold in logic L: then L will satisfy all the usual interpolation and definability properties, together with countable compactness, provided L is reasonably small. The latter assumption can be weakened or removed by using special set-theoretical assumptions. Thus, if Robinson's consistency theorem holds in L, then (i) L is countably compact if its Löwenheim number is $<\mu_0$ = the smallest uncountable measurable cardinal; (ii) if ω is the only measurable cardinal, L is countably compact, or the theories of L characterize every structure up to isomorphism. As a corollary, a partial answer is given to H. Friedman's third problem, by proving that *no* logic L strictly between $L_{\infty\omega}$ and $L_{\infty\infty}$ satisfies interpolation (or Robinson's consistency), unless L-elementary equivalence coincides with isomorphism.

0. Introduction

We study the interplay between compactness and interpolation in soft model theory. We give a partial answer to H. Friedman's third problem in [7], by proving that (if ω is the only measurable cardinal) *no* logic L strictly between $L_{\infty\omega}$ and $L_{\infty\infty}$ obeys the interpolation theorem, unless L-elementary equivalence coincides with isomorphism, (in which case Friedman's problem is still open). The key notion used throughout is Robinson's consistency theorem: its study yields a unified technique to deal both with logics which are small (see Corollaries 5.2 and 5.3 below) and with the most general class of logics in which the sentences of a given type may be a proper class (see Corollary 5.4, Proposition 5.6 and Corollary 5.7, the latter dealing with H. Friedman's third problem). The first sort of results is, incidentally, related with the important theorem stated in the postscriptum of [16]; the second sort of results shows the generality of our methods: in their embryonic stage, the techniques developed for the proof of our main result (Theorem 4.1) also had some algebraic application (see [21]): the latter, in turn, gives a partial answer to Friedman's fourth problem, upon restriction to countable structures (see also [19]).

Theorem 4.1 below, establishing the first connection between notions of compactness and interpolation, states that if logic L has the Robinson property (i.e., if Robinson's consistency theorem holds in L) and for all $\kappa \geq \omega$, L is not

κ-relatively compact, then the theories of L can characterize *every* single-sorted structure up to isomorphism.

Relative compactness was studied by Makowsky and Shelah in [16] and furnished an important tool for investigating the interaction between compactness and descendingly incomplete ultrafilters, mainly in the light of [21], [12] and [9]. The latter papers and results are never used below.

Robinson's consistency theorem was studied (in a slightly different form) in [16]; in the present form it can be found in the postscriptum of [16], as well as in [18–23]; in [18] it is proved that no extension L of $L(Q_1)$ can be countably compact or axiomatizable, if Robinson's consistency theorem holds in L (notice that no special set-theoretical assumptions are involved). In [20], under $V = L$ or $\neg 0^{\#}$ or even $\neg L^{\mu}$, it is proved that full compactness + Craig's interpolation = Robinson's consistency in any extension of first-order logic having a Löwenheim number.

However, this paper is independent of [18–23]. As a matter of fact, the proof of Theorem 4.1, which is the main ingredient of all the other results of this paper, is self-contained: the preparatory lemmas in Section 2 were essentially proved also in [19].

Using Theorem 4.1 (or rather, its Corollary 4.2), in Theorem 5.1 we prove that if ω is the only measurable cardinal, then every logic L having the Robinson property is either countably compact, or else each single-sorted structure is characterized up to isomorphism by its complete theory in L. In case uncountable measurable cardinals exist, and μ_0 is the smallest such cardinal, from the proof of Theorem 5.1 together with Corollary 4.2, one immediately has that either L is countably compact, or else L-equivalence coincides with isomorphism for every single-sorted structure of cardinality below μ_0. In Section 5, Theorem 5.1 is applied *both* to logics subject to smallness conditions, such as having few quantifiers, or having a Löwenheim number $<\mu_0$, *and* to logics where a Löwenheim number may not exist, or the collection of sentences of any given type τ may be a proper class.

Specifically, in Corollary 5.2 we prove that every countably generated logic having the Robinson property automatically satisfies countable compactness, Craig interpolation and all the other usual interpolation and definability properties.

In Corollary 5.3 we prove that if L has the Robinson property and has a Löwenheim number $<\mu_0$, then L is countably compact.

Passing now to applications of Theorem 5.1 to large logics, in Corollary 5.4 we immediately obtain that if ω is the only measurable cardinal and $L \geqslant L(Q_0)$ (where Q_0 is the quantifier 'there are infinitely many'), then either L-elementary equivalence coincides with isomorphism, or else Robinson's consistency theorem does not hold in L. In the above corollary one might equivalently replace $L(Q_0)$ by any logic which is not countably compact. This corollary exhibits a large collection of logics in which Robinson's consistency fails: for example, all L with $L(Q_0) \leqslant L \leqslant L_{\infty\kappa}$, where $\kappa \geqslant \omega$ is an arbitrary cardinal.

Proposition 5.6 establishes the following implication, for all logics $L \geqslant L_{\infty\omega}$:

Craig's interpolation $\rightarrow$ Robinson's consistency.

Notice that no special set theoretical assumptions are involved here.

From Corollary 5.4 and Proposition 5.6 we immediately obtain in Corollary 5.7 the above mentioned result that *no* logic strictly between $L_{\infty\omega}$ and $L_{\infty\infty}$ satisfies interpolation, or Robinson's consistency theorem, unless its theories can characterize every structure up to isomorphism. This theorem gives a partial answer to H. Friedman's third problem in [7]:

"No logic strictly between $L_{\infty\omega}$ and $L_{\infty\infty}$ obeys the interpolation theorem."

Other particular cases of this problem were dealt with by [8], [14] and others.

We assume familiarity with [4], [17] and [6, Sections 1 and 2]; for large logics the reader might consult [ ], though only very simple facts about them are used here.

For further information about Robinson's property, we refer the reader to [16, postscriptum] and to the above mentioned papers by the present author.

In the light of Feferman's remark in [5], that (suitable generalizations of) compactness and interpolation, together with axiomatizability, are the good properties one particularly has in mind when looking for proper extensions of first-order logic, Robinson's consistency theorem is certainly to have a role, both as an interpolation and as a compactness property (see [20]): thus, for example, H. Friedman's *fourth* problem in [7] can be rephrased with no substantial loss as the problem of finding proper extensions of first-order logic having Robinson's property and a Löwenheim number (compare with the first problem in the introduction of [16]).

On the other hand, our results in Section 5 in this paper show that Robinson's property is a very strong requirement on logics; in particular, Corollaries 5.2 and 5.3 suggest that if there exists any *small* logic $L > L_{\omega\omega}$ having Robinson's property, then L will share with $L_{\omega\omega}$ many of its good properties, and the extra expressive power of L is not to be easily found: in fact, by Theorem 5.1 and a well-known result due to Lindström, no sublogic of $L_{\infty\omega}$ can be of any help to this search; further, one cannot hope that $L \geqslant L(Q_1)$ by the above mentioned result in [18].

The situation does not look better if we relax the bounds on the size of L: in fact, Corollary 5.4 exhibits a lot of *large* logics in which Robinson's theorem fails (if ω is the only measurable cardinal).

From the algebraic viewpoint (i.e. by only considering L-elementary equivalence $\equiv_L$), in [21] it is proved that, upon restriction to countable structures of finite type, the only non-pathological equivalence relations satisfying Robinson's consistency theorem are elementary equivalence and isomorphism.

The problems involved in the search of good equivalence relations on structures in the general case, and the corresponding—more difficult and fundamental—problems concerning good extensions of first-order logic, can only be solved by means of (i) a deeper study of the sensitivity of soft model theory to specific set-theoretical axioms, and (ii) the introduction of powerful techniques allowing a unified treatment of both logics which are small and infinitary logics.

Concerning (ii), the development of the tools introduced in this paper to prove Theorem 4.1, and centered around the notion of Robinson's consistency, might bring some profit.

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1. Preliminaries

Throughout this paper τ, τ', and τ'' will denote (many-sorted, similarity) types, i.e. *sets* of sorts and symbols as defined in [4]; we shall not require that types are finite. $\mathrm{Str}(\tau)$ is the *class* of all structures of type τ, each structure $\mathfrak{A} \in \mathrm{Str}(\tau)$ being a function as defined in [4]; notice that the universe A of $\mathfrak{A}$ is always assumed to be a *set*, whose cardinality is denoted by $|A|$; A, B and N are the universes of $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{N}$ respectively. If $B' \subseteq B$ and B' is nonempty on each sort of $\mathfrak{B}$, then $\mathfrak{B} \mid B'$ is the substructure of $\mathfrak{B}$ generated by B'.

A *logic* L is an ordered pair $(\mathrm{Stc}_L, \models_L)$ satisfying the axioms of *occurrence*, *expansion*, *renaming* and *isomorphism*, and which is *closed under the first-order operations* (see [4, pp. 155–157; 1, p. 234]). Note that in this paper *all* structures and *all* types are admitted; notice also that $\mathrm{Stc}_L(\tau)$ *need not be a set* and there need not exist any upper bound for the cardinality of the set of symbols occurring in the sentences of L. If $\varphi \in \mathrm{Stc}_L(\tau)$, then we simply say that φ is of type τ. Formulas are dealt with exactly as in [4, p. 156]. In our logics L we allow *relativization* of formula φ to formula $\psi(x, z_1, \ldots, z_r)$, where the z's act as parameters: the result of this process is still a formula of L, written

$$\varphi^{\{x \mid \psi(x, z_1, \ldots, z_r)\}}.$$

See [6] for the relationship between relativization and substructures. Examples L of logics are $L_{\infty\kappa}$ for $\kappa \geq \omega$ a cardinal, and $L_{\infty\infty}$ (see [3]); here, $\mathrm{Stc}_L(\tau)$ is a proper class and in the sentences of L arbitrarily large sets of symbols (and sorts) occur. An important subclass of logics have the form $L = L_{\omega\omega}(Q^i)_{i \in I}$ (see [17]): here $\mathrm{Stc}_L(\tau)$ is a set, provided I is a set; in case $|I| \leq \omega$, i.e. L has countably many quantifiers, we say that L is countably generated: $L_{\omega\omega}$ is first-order logic and $L(Q_0) = L_{\omega\omega}(Q_0)$ is the logic with quantifier "there exist infinitely many" (see [15] or [17]). For $\mathfrak{A} \in \mathrm{Str}(\tau)$, $\mathrm{th}_L \mathfrak{A}$ is the class of all sentences φ of type τ such that $\mathfrak{A} \models_L \varphi$; we often drop subscript L, if no confusion may arise; for $T \subseteq \mathrm{Stc}_L(\tau)$ a theory in L, $\mathrm{mod}_L T$ is the class of those structures $\mathfrak{A}$ such that $\mathfrak{A} \models_L T$. Given structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \equiv_L \mathfrak{B}$ means that $\mathrm{th}_L \mathfrak{A} = \mathrm{th}_L \mathfrak{B}$ (read: $\mathfrak{A}$ and $\mathfrak{B}$ are L-*elementarily equivalent*). Given two logics L and L', $L <_{\mathrm{Th}} L'$ means that whenever $\mathfrak{A} \equiv_{L'} \mathfrak{B}$ then also $\mathfrak{A} \equiv_L \mathfrak{B}$ (i.e., L'-elementary equivalence is finer than L-elementary equivalence); $L \leq L'$ means that for each type τ and $\varphi \in \mathrm{Stc}_L(\tau)$

there is $\varphi' \in \mathrm{Stc}_{L'}(\tau)$ with the same models. Following [17], we let $I\mathfrak{A} = \{\mathfrak{B} \mid \mathfrak{B} \cong \mathfrak{A}\}$.

For the notion of (κ, λ)-compactness, see [17] (the definition is the same even for the more general class of logics considered in this paper); countable compactness means (ω, ω)-compactness; full compactness (or, shortly, compactness) is (κ, ω)-compactness for all $\kappa \geqslant \omega$.

In their paper [16] Makowsky and Shelah studied an important related notion, given by the following definition (compare with [16, definition after the statement of Theorem 6.2]):

Definition 1.1. We say that L is *κ-relatively compact* (for short: κ-r.c.), for κ a cardinal $\geqslant \omega$, iff for any classes of sentences Σ and Γ with $|\Sigma| = \kappa$, if for all $\Sigma_0 \subseteq \Sigma$ with $|\Sigma_0| < k$, $\Sigma_0 \cup \Gamma$ is consistent, then $\Sigma \cup \Gamma$ is consistent (i.e., $\Sigma \cup \Gamma$ has a model).

For the notion of Craig interpolation property, Δ-closure, (weak) Beth property, see [17]; an important related notion, studied extensively in [19], is given by the following definition (compare with [11] for the first-order case):

Definition 1.2. We say that in L *Robinson's consistency theorem* holds (or, *L has the Robinson property*) iff given any types τ, τ' and τ'', and classes of sentences T, T' and T'', if T is complete in τ and T' and T'' are consistent extensions of T in type τ' and τ'' respectively, with $\tau = \tau' \cap \tau''$, then $T' \cup T''$ is consistent.

This is easily seen to be equivalent to saying that $\forall \mathfrak{A}' \in \mathrm{Str}(\tau')$, $\forall \mathfrak{A}'' \in \mathrm{Str}(\tau'')$, if $\tau = \tau' \cap \tau''$ and $\mathfrak{A}' \restriction \tau \equiv_L \mathfrak{A}'' \restriction \tau$, then $\exists \mathfrak{B} \in \mathrm{Str}(\tau' \cup \tau'')$ such that $\mathfrak{B} \restriction \tau' \equiv_L \mathfrak{A}'$ and $\mathfrak{B} \restriction \tau'' \equiv_L \mathfrak{A}''$. (Compare with [10] for the case of $L_{\omega\omega}$.)

Following [17, p. 161 and p. 175] we give the following:

Definition 1.3. We say that logic L satisfies the *Löwenheim–Skolem* (L–S) *theorem* for cardinal κ iff every sentence ψ of L which has a model, has a model of cardinality $\leqslant \kappa$; the *Löwenheim number* of L is the smallest cardinal κ such that the L–S theorem holds for κ in L; logic L satisfies the L–S theorem for cardinal κ *for theories of cardinality* $\leqslant \lambda$ iff each consistent theory T of L with $|T| \leqslant \lambda$ has a model $\mathfrak{A}$ with $|\mathfrak{A}| \leqslant \kappa$.

Löwenheim-type requirements, as well as countable generatedness, impose restrictions on the size of logics. Under such restrictions we shall prove below that the Robinson property (i.e. an interpolation and definability property) implies countable compactness; restrictions about the size of logic L can be removed by introducing special set-theoretical assumptions. Sections 2, 3 and 4 below establish some important facts about the Robinson property, which are subsequently used for the study of the relationship between compactness and interpolation properties both in small and in huge logics.

2.

Proposition 2.1. *Let in L Robinson's consistency theorem hold; let* $\langle\mu,<\rangle$ *be an infinite ordinal and assume that* $\mathrm{mod}_L\ \mathrm{th}_L\langle\lambda,<\rangle = I\langle\lambda,<\rangle$ *for all ordinals* $\langle\lambda,<\rangle \leq \langle\mu,<\rangle$. *Then we have* $\mathrm{mod}_L\ \mathrm{th}_L\mathfrak{A} = I\mathfrak{A}$ *for every single-sorted structure* $\mathfrak{A}$ *with* $|\mathfrak{A}| \leq |\mu|$.

For the proof we prepare:

Lemma 2.2. *Under the hypotheses of Proposition* 2.1 *we have* $\mathrm{mod}_L\ \mathrm{th}_L\kappa = I\kappa$ *for any cardinal* $\kappa \leq |\mu|$.

Remark. Notice that $\mathrm{th}_L\kappa$ is a theory in the pure identity language of L.

Proof. Assume $\nu \equiv_L \kappa$ and $|\nu| \neq \kappa$; it is no loss of generality to assume that $|\nu| = \nu$; let $\langle\kappa,<\rangle$ denote the least ordinal of cardinality κ.

Case 1: $\nu > \kappa$; then consider expansion

$$\nu' = \langle\nu, c_\varepsilon\rangle_{\varepsilon<\nu} \quad \text{(for } \varepsilon \text{ an ordinal).}$$

By hypothesis

$$\mathrm{th}_L\,\nu' \cup \mathrm{th}_L\langle\kappa,<\rangle \supseteq \mathrm{th}_L\,\kappa$$

is inconsistent; but this contradicts the assumed Robinson property of L.

Case 2: $\nu < \kappa$; let $\langle\nu,<\rangle$ be the least ordinal of cardinality ν and let $\mathrm{th}_L\langle\kappa,<'\rangle$ be the complete theory of $\langle\kappa,<\rangle$ in type $\{<'\}$ (with $<'$ a binary relation symbol different from $<$); then

$$\mathrm{th}_L\langle\kappa,<'\rangle \cup \mathrm{th}_L\langle\nu,<\rangle \supseteq \mathrm{th}_L\,\kappa$$

is inconsistent by hypothesis, again contradicting the assumed Robinson property of L. $\square$

Lemma 2.3. *Under the hypotheses of Proposition* 2.1, *we have that* $\mathrm{mod}_L\ \mathrm{th}_L\mathfrak{N} = I\mathfrak{N}$ *for every single-sorted expansion* $\mathfrak{N} = \langle\nu,<,\ldots\rangle$ *of each ordinal* $\langle\nu,<\rangle \leq \langle\mu,<\rangle$.

Proof. Let $\mathfrak{M}, \mathfrak{N} \in \mathrm{Str}(\tau)$ with $\mathfrak{M} \equiv_L \mathfrak{N} = \langle\nu,<,\ldots\rangle$; by hypothesis and by reduct, $\mathfrak{M}\restriction\{<\} \cong \mathfrak{N}\restriction\{<\}$; we assume without loss of generality that $\mathfrak{M}\restriction\{<\} = \mathfrak{N}\restriction\{<\}$;

assume $\mathfrak{M} \not\equiv \mathfrak{N}$ (absurdum hypothesis); then for some ordinal $\varepsilon < \nu$ and for some symbol $R \in \tau$ we have $\mathfrak{M} \models \neg R\varepsilon$ and $\mathfrak{N} \models R\varepsilon$ (or vice versa); notice that we have assumed R unary relation for simplicity; in the general case one shall work with r-tuplas of ordinals smaller than ν. Let $T' = \mathrm{th}_L\langle\mathfrak{N}, \varepsilon\rangle$ of type $\tau \cup \{n'\}$ and $T'' = \mathrm{th}_L\langle\mathfrak{M}, \varepsilon\rangle$ of type $\tau \cup \{n''\}$. We claim that $T' \cup T''$ is inconsistent; if not, let $\mathfrak{S} \models T' \cup T''$; then $\mathfrak{S} \restriction \{<\} \cong \langle\nu, <\rangle$ by hypothesis, and $\mathfrak{S} \models n' = n''$ by the assumed characterizability of $\varepsilon < \nu \leq \mu$ (one has to relativize $\mathrm{th}_L\langle\varepsilon, <\rangle$ to $\{x \mid x < n'\}$ and to $\{x \mid x < n''\}$; then one notes that these two relativized theories are contained in T' and T'' respectively); further, $\mathfrak{S} \models Rn' \wedge \neg Rn''$, which is clearly impossible. Having proved our claim, we have also exhibited two consistent extensions T' and T'' of $\mathrm{th}_L \mathfrak{N}$ which are counterexamples to the assumed Robinson property of L. □

Proof of Proposition 2.1. Let $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$, $|\mathfrak{A}| \leq \mu$ with $\mathfrak{B} \equiv_L \mathfrak{A}$; by Lemma 2.2 $|\mathfrak{B}| = |\mathfrak{A}| = \nu \leq \mu$; let $\langle\nu, <\rangle$ be the least ordinal of cardinality ν; let $\mathfrak{B}' = \langle\mathfrak{B}, <'\rangle$ where $<'$ well-orders B with order type $\langle\nu, <\rangle$; similarly let $\mathfrak{A}' = \langle\mathfrak{A}, <\rangle$ where $<$ has order type $\langle\nu, <\rangle$; by the Robinson property, $\mathrm{th}_L \mathfrak{A}' \cup \mathrm{th}_L \mathfrak{B}'$ is consistent, i.e. there exists $\mathfrak{M} \in \mathrm{Str}(\tau \cup \{<, <'\})$ with

$$\mathfrak{M} \restriction \tau \cup \{<'\} \equiv_L \mathfrak{B}',$$

$$\mathfrak{M} \restriction \tau \cup \{<\} \equiv_L \mathfrak{A}'.$$

By Lemma 2.3, noting that $\mathfrak{B}'$ and $\mathfrak{A}'$ are (isomorphic to) expansions of well-ordered structures of order type $\leq \langle\mu, <\rangle$ we have

$$\mathfrak{M} \restriction \tau \cup \{<'\} \cong \mathfrak{B}',$$

$$\mathfrak{M} \restriction \tau \cup \{<\} \cong \mathfrak{A}'.$$

By reduct

$$\mathfrak{M} \restriction \tau \cong \mathfrak{B}$$

and

$$\mathfrak{M} \restriction \tau \cong \mathfrak{A},$$

i.e.

$$\mathfrak{A} \cong \mathfrak{B}. \quad \square$$

3.

Proposition 3.1. *Let in L Robinson's consistency theorem hold; let κ be an infinite cardinal and $\langle\kappa, <\rangle$ the least ordinal of cardinality κ; assume that L is not κ-r.c. and that $\mathrm{mod}_L\, \mathrm{th}_L\langle\lambda, <\rangle = I\langle\lambda, <\rangle$ for all ordinals $\langle\lambda, <\rangle < \langle\kappa, <\rangle$. Then $\mathrm{mod}_L\, \mathrm{th}_L\langle\kappa, <\rangle = I\langle\kappa, <\rangle$.*

Proof. By assumption there exist theories Σ with $|\Sigma| = \kappa$, say, $\Sigma = \{\varphi_\varepsilon \mid \varepsilon < \kappa\}$ and Γ such that $\Sigma \cup \Gamma$ is inconsistent but $\forall \Sigma' \subseteq \Sigma$ with $|\Sigma'| < \kappa$ $\Sigma' \cup \Gamma$ is consistent; let

τ_0 be the type of $\Sigma \cup \Gamma$ and assume, without loss of generality, that τ_0 has only relation symbols; for any ordinal $\alpha < \kappa$ let $\Sigma_\alpha = \{\varphi_\varepsilon \mid \varepsilon < \alpha\}$ and let $\mathfrak{A}_\alpha$ be such that

$$\mathfrak{A}_\alpha \models \Sigma_\alpha \cup \Gamma.$$

Without loss of generality, assume that $A_\beta \cap A_\gamma = \emptyset$ whenever $\beta \neq \gamma$; expand τ_0 to τ by adding one new sort and new symbols

$$\mathrm{g}, <, c_\alpha \quad \text{(for all ordinals } \alpha < \kappa)$$

where g is a unary function, $<$ is a binary relation and the c_α's are constants. Let $\mathfrak{M} \in \mathrm{Str}(\tau)$ have the following properties:

$$\mathfrak{M} \restriction \{<, c_\alpha\}_{\alpha<\kappa} = \langle \kappa, <, \alpha \rangle_{\alpha<\kappa},$$

range of $\mathrm{g} = \kappa$,

domain of g = union of the universes of the $\mathfrak{A}_\alpha$'s,

$\mathrm{g}^{-1}(\alpha) =$ universe of $\mathfrak{A}_\alpha$, $A_\alpha \quad (\alpha < \kappa)$,

$$\mathfrak{M} \restriction \tau_0 \mid \mathrm{g}^{-1}(\alpha) = \mathfrak{A}_\alpha \quad (\alpha < \kappa).$$

Then $\mathfrak{M}$ can be imagined as a 'disjoint union' of the $\mathfrak{A}_\alpha$'s equipped with an index function g for their (pairwise disjoint) universes. $\mathfrak{M}$ satisfies theory T of type τ given by the following axioms (here we use symbol μ for a variable over the new sort):

$(1)_{\alpha<\beta<\kappa}$

$<$ is a linear ordering over the last sort and $c_\alpha < c_\beta$,

$(2)_{\alpha<\kappa}$

the order type of the set of predecessors of c_α is $\langle \alpha, < \rangle$

(this is expressed by relativizing to $\{\mu \mid \mu < c_\alpha\}$ theory $\mathrm{th}_L\langle \alpha, <\rangle$ which is assumed to characterize α up to isomorphism),

$(3)_{i \in \text{sorts of } \tau_0}$

$$\forall \mu \, \exists x^i \, \mathrm{g}(x^i) = \mu$$

(so that each inverse image $\mathrm{g}^{-1}(\mu)$ is non-empty on each sort i of τ_0).

$(4)_{\alpha<\kappa}$

$$\forall \mu (c_\alpha < \mu \rightarrow \varphi_\alpha^{\{x \mid \mathrm{g}(x) = \mu\}})$$

(so that Σ_μ is satisfied in the μth inverse image of g),

$(5)_{\psi \in \Gamma}$

$$\forall \mu \, \psi^{\{x \mid \mathrm{g}(x) = \mu\}}$$

(so that also Γ is satisfied in the μth inverse image of g).

Claim 1. *In any model $\mathfrak{B} \models T$, the c_α's are unbounded, i.e.*

$$\bar{\exists} b \in B \text{ such that } \mathfrak{B} \models c_\alpha < b \quad \forall \alpha < \kappa.$$

As a matter of fact, if such $\mathfrak{B}$ and b exist, then $\langle \mathfrak{B}, b \rangle$ has the following properties:

$g^{-1}(b) \neq \emptyset$ on each sort of τ_0 by (3),

$\forall \alpha < b$, φ_α holds in $\mathfrak{B} \upharpoonright \tau_0 \mid g^{-1}(b)$ by (4),

Γ holds in $\mathfrak{B} \upharpoonright \tau_0 \mid g^{-1}(b)$ by (5),

so that $\Sigma \cup \Gamma$ is consistent, a contradiction.

Claim 2. $\forall \mathfrak{B} \models T$ $\mathfrak{B} \upharpoonright \{<\} \cong \langle \kappa, < \rangle$.

As a matter of fact, by (1) $|\mathfrak{B} \upharpoonright \{<\}| \geqslant \kappa$; $\mathfrak{B} \upharpoonright \{<\}$ is well-ordered, for otherwise there is an infinitely descending chain $a_0 > a_1 > \cdots$; by Claim 1, $\exists \lambda < \kappa$ such that $c_\lambda > a_0$ in $\mathfrak{B}$, so that the set of precedessors of c_λ is not well-ordered, contradicting (2) in view of the assumption about $\text{th}_L \langle \lambda, < \rangle$. Similarly, any $a \in B$ has strictly less than κ many predecessors, for otherwise, by Claim 1, $\exists \lambda < \kappa$ such that $c_\lambda > a$, and c_λ would not satisfy (2).

Having proved Claim 2, we complete the proof of Proposition 3.1 as follows: assume $\mathfrak{N} \equiv_L \langle \kappa, < \rangle$ and $\mathfrak{N} \not\cong \langle \kappa, < \rangle$ (absurdum hypothesis); then

(a) either $\mathfrak{N}$ is not well-ordered,

(b) or for some $n \in N$ the cardinality of the set of predecessors of n is $\geqslant \kappa$,

(c) or, finally, $|\mathfrak{N}| < \kappa$.

Taking care of (a): if $\mathfrak{N}$ is not well-ordered, let $a_0 > a_1 > \cdots$ be an infinitely descending chain in N; expand $\mathfrak{N}$ to $\mathfrak{N}' = \langle \mathfrak{N}, a_0, a_1, \ldots \rangle$ and notice that $\mathfrak{N}' \models T'$, where T' is defined by

$$T' = \text{th}_L \langle \kappa, < \rangle \cup \{a_0 > a_1 \wedge a_1 > a_2 \wedge \cdots\}.$$

On the other hand, by Claim 2, the $\{<\}$-reduct of any model of T is well ordered, hence the union of theories T' and $T \cup \text{th}_L \langle \kappa, < \rangle$ is inconsistent, both theories being consistent extensions of the complete theory $\text{th}_L \langle \kappa, < \rangle$ in such a way that Robinson's consistency theorem does not hold in L, a contradiction.

Taking care of (b); let some $n \in N$ have, say κ many predecessors, $\{d_\lambda\}_{\lambda < \kappa}$; then by a similar argument to (a) above, we see that a contradiction can be drawn from the assumed Robinson property of L and the inconsistency of the union of theories $T \cup \text{th}_L \langle \kappa, < \rangle$ and $\text{th}_L \langle \mathfrak{N}, n, d_\lambda \rangle_{\lambda < \kappa}$, both being consistent extensions of the complete theory $\text{th}_L \langle \kappa, < \rangle$, and having no common symbols except $<$.

Taking care of (c): let $|\mathfrak{N}| = \zeta$ with $\zeta < \kappa$; expand $\mathfrak{N}$ to $\mathfrak{N}' = \langle \mathfrak{N}, <' \rangle$ where $<'$ well-orders N with order type $\langle \zeta, < \rangle$ (viz. the least ordinal of cardinality ζ); by assumption, if $\mathfrak{B} \models \text{th}_L \mathfrak{N}'$ then $\mathfrak{B} \upharpoonright \{<'\} \cong \langle \zeta, < \rangle$; this shows in particular that $|\mathfrak{B}| = \zeta$, hence the union of theories $\text{th}_L \mathfrak{N}'$ and $T \cup \text{th}_L \langle \kappa, < \rangle$ is inconsistent, both theories being consistent extensions of the complete theory $\mathrm{th}_L\langle\kappa, <\rangle$ in such a way that Robinson's consistency theorem does not hold in L, a contradiction.

This concludes the proof of Proposition 3.1. □

Corollary 3.2. *Let in L Robinson's consistency theorem hold; assume that L is not ω-r.c.; then* $\mathrm{mod}_L\ \mathrm{th}_L\langle\omega, <\rangle = I\langle\omega, <\rangle$.

Proof. Immediate from Proposition 3.1 by observing that all finite ordinals are characterizable up to isomorphism in $L_{\omega\omega} \leqslant L$. □

4.

We now draw some consequences from Propositions 2.1 and 3.1, relating the expressive power of logic L to the class of cardinals κ such that L is not κ-r.c.

Theorem 4.1. *Let L have the Robinson property; assume that for every infinite cardinal κ, L is not κ-r.c.; then for any single-sorted structure $\mathfrak{A}$ we have*

$$\mathrm{mod}_L\ \mathrm{th}_L\mathfrak{A} = I\mathfrak{A}.$$

Proof. By induction on $|\mathfrak{A}| \geqslant \omega$ (the finite case is trivial):

Initial step: $|\mathfrak{A}| = \omega$. Notice that $\mathrm{mod}_L\ \mathrm{th}_L\langle\omega, <\rangle = I\langle\omega, <\rangle$ by Corollary 3.2. Also, each finite ordinal is characterizable up to isomorphism in $L \geqslant L_{\omega\omega}$; now apply Proposition 2.1.

Induction step: let $\langle\alpha, <\rangle$ be the least ordinal of cardinality $|\mathfrak{A}|$; given any ordinal $\langle\beta, <\rangle < \langle\alpha, <\rangle$ we have that $\mathrm{mod}_L\ \mathrm{th}_L\langle\beta, <\rangle = I\langle\beta, <\rangle$ by induction hypothesis; then $\mathrm{mod}_L\ \mathrm{th}_L\langle\alpha, <\rangle = I\langle\alpha, <\rangle$ by Proposition 3.1 and $\mathrm{mod}_L\ \mathrm{th}_L\mathfrak{A} = I\mathfrak{A}$ now follows from Proposition 2.1. □

Corollary 4.2. *Let L have Robinson's property, and let κ_0 be an infinite cardinal; assume that for all infinite cardinals $\kappa \leqslant \kappa_0$, L is not κ-r.c.; then for every single-sorted structure $\mathfrak{A}$ with $|\mathfrak{A}| \leqslant \kappa_0$, we have* $\mathrm{mod}_L\ \mathrm{th}_L\mathfrak{A} = I\mathfrak{A}$.

Proof. By truncating the induction argument of Theorem 4.1 after κ_0. □

5.

Theorem 5.1. (Assuming ω is the only measurable cardinal.) *Let in L Robinson's consistency theorem hold; then*

(i) *either L is countably compact,*

(ii) *or, for any two single-sorted structures* $\mathfrak{A}$ *and* $\mathfrak{B}$ *we have*

$\mathfrak{A} \equiv_L \mathfrak{B}$ *iff* $\mathfrak{A} \cong \mathfrak{B}$.

Proof. Assume L is not countably compact; to prove the theorem, by Theorem 4.1, it suffices to prove that L is not κ-r.c. $\forall \kappa \geq \omega$. As a matter of fact, if this were not the case, let θ be the least cardinal such that L is θ-r.c.; notice that $\theta > \omega$, otherwise L would also be countably compact. By Corollary 4.2 we have

$$\operatorname{mod}_L \operatorname{th}_L\langle\beta, <\rangle = I\langle\beta, <\rangle \quad \forall \beta < \theta. \tag{1}$$

Define the single-sorted type τ by adding a symbol for each $\alpha < \theta$, $X \subseteq \theta$, $f : \theta \to \theta$, i.e.,

$$\tau = \{<, \alpha, U_X, f\}_{\alpha<\theta,\, X\subseteq\theta,\, f:\theta\to\theta}, \tag{2}$$

and structure $\mathfrak{A} \in \operatorname{Str}(\tau)$ by:

$$\mathfrak{A} = \langle \theta, <, \alpha, X, f\rangle_{<\theta,\, X\subseteq\theta,\, f:\theta\to\theta}, \tag{3}$$

so that $\alpha^{\mathfrak{A}} = \alpha$, $f^{\mathfrak{A}} = f$, $U_X^{\mathfrak{A}} = X$. Let $\Gamma = \operatorname{th}_L \mathfrak{A}$, and $\Sigma = \{b > \alpha \mid \alpha < \theta\}$, for b a new constant symbol; let $\mathfrak{B} \models \Sigma \cup \Gamma$, such $\mathfrak{B}$ existing by the assumed θ-relative compactness of L; in particular

$$\mathfrak{B} \restriction \tau \equiv_L \mathfrak{A}. \tag{4}$$

Let $\mathfrak{D} \subseteq P(\theta)$ be a collection of subsets of θ given by

$$Y \in \mathfrak{D} \quad \text{iff } \mathfrak{B} \models_L U_Y(b), \tag{5}$$

(compare with [16, 6.4(ii)]). By (1)–(5) one sees that $\mathfrak{D}$ is a nonprincipal ultrafilter on θ.

Claim. $\mathfrak{D}$ *is* θ*-complete.*

Proof. If not, $\mathfrak{D}$ is μ-d.i. for some $\mu < \theta$, so let $\mathfrak{D}' = \{Y_\alpha\}_{\alpha<\mu}$ be a descending chain of elements of $\mathfrak{D}$ of length μ with $Y = \bigcap_{\alpha<\mu} Y_\alpha \notin \mathfrak{D}$ i.e., without loss of generality, $Y = \emptyset$. Let $g : \theta \to \mu$ be given by

$$g(\alpha) = \beta \quad \text{iff } \alpha \in Y_\beta \setminus Y_{\beta+1}\ (\alpha < \theta, \beta < \mu). \tag{6}$$

By (2) and (3), function g is denoted by g itself in $\mathfrak{A}$; let similarly $U_\alpha \in \tau$ be the only unary relation symbol such that $U_\alpha^{\mathfrak{A}} = Y_\alpha$ $(\alpha < \mu)$. Now we have:

$$\forall \alpha < \mu \ \ \mathfrak{A}, \mathfrak{B} \models_L \forall x (g(x) \leq \alpha \to \neg U_{\alpha+1}(x)) \quad \text{by (4) and (6)},$$

$$\forall \alpha < \mu \ \ \mathfrak{B} \models_L g(b) > \alpha, \quad \text{as } \mathfrak{B} \models_L U_{\alpha+1}(b) \text{ by (5)},$$

$$\mathfrak{B} \models_L \forall \alpha < \mu \ \ g(b) > \alpha, \quad \text{by (1) and (4)},$$

(characterizability of $\mu < \theta$); thus, we conclude that

$$\mathfrak{B}, \mathfrak{A} \models_L \exists x (\forall \alpha < \mu \ \ g(x) > \alpha), \quad \text{by (4)},$$

so that $Y \neq \emptyset$, and the claim is proved by reductio ad absurdum. □

Having proved our claim, we see that $\theta > \omega$ is measurable, which contradicts our set-theoretical hypotheses; therefore L cannot be θ-r.c. $\square$

Theorem 5.1 can be applied to 'small' logics as follows:

Corollary 5.2. *Let in $L = L_{\omega\omega}(Q^i)_{i<\omega}$ Robinson's consistency theorem hold. Then we have*:

(i) *L is countably compact*;

(ii) *L has Craig's interpolation, Δ-closure and (weak) Beth property.*

Proof. (i) Immediate from Theorem 5.1, if there are no uncountable measurable cardinals, upon noting that there are no more than 2^ω complete theories in the pure identity language of any countably generated logic L. On the other hand, if μ_0 is the smallest uncountable measurable cardinal, then by inspection of the proof of Theorem 5.1 one sees that if L is not countably compact (absurdum hypothesis), then L is not κ-r.c. for each infinite $\kappa < \mu_0$. Then using Corollary 4.2 we have that each single-sorted structure $\mathfrak{A}$ with $|A| < \mu_0$ is characterizable up to isomorphism by its complete theory in L. Thus the number of complete theories in the pure identity language of L is $>2^\omega$, which contradicts the assumption about L being countably generated.

(ii) By (i), L is countably compact; this, together with L being countably generated and having Robinson's property is well known to imply Craig's interpolation property (see [16, 2.1]). The other interpolation and definability properties now follow from Craig's interpolation property, by well-known results in soft model theory (see [17] or [16]). $\square$

Corollary 5.3. *Let in L Robinson's consistency theorem hold. Assume the Löwenheim number λ of L exists and either λ is strictly smaller than the first uncountable measurable cardinal μ_0, or there are no uncountable measurable cardinals at all. Then L is countably compact.*

Proof. By a familiar argument (see [16, 1.6]) there are at most κ many inequivalent sentences in the pure identity language of L, where

$$\kappa = 2^{2^\lambda},$$

hence there are at most 2^κ inequivalent theories in the pure identity language of L. On the other hand, by arguing as in the proof of Corollary 5.2 in the light of Theorem 5.1 and Corollary 4.2, we have that, if L is not countably compact (absurdum hypothesis), then there are at least μ_0 many inequivalent complete theories in the pure identity language of L, and $\mu_0 > 2^\kappa$ since $\lambda < \mu_0$ and μ_0 is well known to be strongly inaccessible. (If no uncountable measurable cardinals exist, the argument becomes trivial.) $\square$

Theorem 5.1 can be applied to 'large' logics as follows:

Corollary 5.4. (*Assuming ω is the only measurable cardinal.*) *Let L be such that $L(Q_0) \leqslant L$; then either $\equiv_L = \cong$, or else Robinson's consistency theorem does not hold in L.*

Proof. Since $L(Q_0)$ is not countably compact, then so is L; now, if Robinson's theorem holds in L, then one simply applies Theorem 5.1 above to get the desired conclusion. $\square$

Remark 5.5. Corollary 5.4 exhibits a very large class of logics in which Robinson's consistency theorem fails: for example, all L with $L(Q_0) \leqslant L \leqslant L_{\infty\kappa}$, where $\kappa \geqslant \omega$; actually the same holds by replacing $L(Q_0)$ by any logic which is not countably compact.

Proposition 5.6. *If $L \geqslant L_{\infty\omega}$ satisfies Craig's interpolation, then L satisfies Robinson's consistency theorem.*

Proof. Assume Robinson's consistency fails in L; then

$$\exists \mathfrak{A}' \in \mathrm{Str}(\tau'), \exists \mathfrak{A}'' \in \mathrm{Str}(\tau'') \text{ such that, letting } \tau = \tau' \cap \tau'', \text{ we have that } \mathfrak{A}' \restriction \tau \equiv_L \mathfrak{A}'' \restriction \tau, \text{ but } \bar{\exists} \mathfrak{D} \in \mathrm{Str}(\tau' \cup \tau'') \text{ such that } \mathfrak{D} \restriction \tau' \equiv_L \mathfrak{A}' \text{ and } \mathfrak{D} \restriction \tau'' \equiv_L \mathfrak{A}''. \tag{1}$$

Let now $\tau_{A'}$ be the diagram type (also called diagram language) of $\mathfrak{A}'$ (see [11, p. 57]), and let $\mathfrak{A}'_{A'}$ be the diagram expansion of $\mathfrak{A}'$; let $D(\mathfrak{A}')$ be the diagram of $\mathfrak{A}'_{A'}$, i.e. the set of all atomic and negated atomic sentences of type $\tau_{A'}$ which are true in $\mathfrak{A}'_{A'}$; recall the basic fact that universes of structures are sets. It is well known that structure $\mathfrak{A}'_{A'}$ can be characterized up to isomorphism by the sentence $\varphi' \in \mathrm{Stc}_{L_{\infty\omega}}(\tau_{A'})$ given by

$$\bigwedge D(\mathfrak{A}') \wedge \forall x \bigvee_{a' \in A'} a' = x,$$

(see [11, p. 96]); similarly, $\mathfrak{A}''_{A''}$ is characterized by the sentence $\varphi'' \in \mathrm{Stc}_{L_{\infty\omega}}(\tau_{A''})$ given by

$$\bigwedge D(\mathfrak{A}'') \wedge \forall x \bigvee_{a'' \in A''} x = a''.$$

Notice that both φ' and φ'' are also sentences of L; without loss of generality we assume

$$\tau = \tau' \cap \tau'' = \tau_{A'} \cap \tau_{A''}.$$

Claim 1. $\emptyset = \mathrm{mod}_L \varphi' \cap \mathrm{mod}_L \varphi''$.

Proof. Otherwise, let $\mathfrak{M} \in \mathrm{Str}(\tau_{A'} \cup \tau_{A''})$ be such that $\mathfrak{M} \models \varphi' \wedge \varphi''$ (absurdum hypothesis); then $\mathfrak{M} \restriction \tau_{A'} \models \varphi'$ hence $\mathfrak{M} \restriction \tau_{A'} \cong \mathfrak{A}'_{A'}$, by the assumption about φ', whence $\mathfrak{M} \restriction \tau' \cong \mathfrak{A}'$ by reduct; by isomorphism property of logic L (see [6] or [1]), it now follows that

$$\mathfrak{M} \restriction \tau' \equiv_L \mathfrak{A}'.$$

Similarly we get

$$\mathfrak{M} \restriction \tau'' \equiv_L \mathfrak{A},$$

thus contradicting (1) above. □

Claim 2. $\bar{\exists} \varphi \in \mathrm{Stc}_L(\tau)$ *such that* $\mathrm{mod}_L \varphi' \subseteq \mathrm{mod}_L \varphi$ *and* $\mathrm{mod}_L \varphi'' \subseteq \mathrm{mod}_L \neg\varphi$.

Proof. Otherwise, let φ be a counterexample; then from $\mathfrak{A}'_{A'} \models \varphi'$ we get $\mathfrak{A}'_{A'} \models \varphi$, hence $\mathfrak{A}' \models \varphi$ (as $\mathfrak{A}'_{A'} \restriction \tau' \cong \mathfrak{A}'$ and φ is of type τ); similarly, $\mathfrak{A}'' \models \neg\varphi$, thus contradicting the assumption in (1) that $\mathfrak{A}' \restriction \tau \equiv_L \mathfrak{A}'' \restriction \tau$. □

We have thus exhibited two sentences, φ' of type $\tau_{A'}$, and φ'' of type $\tau_{A''}$ having no common model and such that no sentence φ of type $\tau_{A'} \cap \tau_{A''}$ in logic L can 'separate' φ' and φ'', in the sense of Claim 2 above; this shows that interpolation fails and proves our result about L. □

By combining Corollary 5.4 and Proposition 5.6 together, we immediately get the following partial answer to H. Friedman's third problem in [7]: "No logic strictly between $L_{\infty\omega}$ and $L_{\infty\infty}$ obeys the interpolation theorem":

Corollary 5.7. (*Assuming ω is the only measurable cardinal.*) *No logic L satisfying the condition $L \geqslant L_{\infty\omega}$ obeys the interpolation theorem, unless $L_{\infty\infty} <_{\mathrm{Th}} L$.*

Remark 5.8. Thus, no logic between $L_{\infty\omega}$ and $L_{\infty\infty}$ obeys the interpolation theorem unless its complete theories are able to characterize every structure up to isomorphism, in which case Friedman's problem is still open. For similar results dealing with such particular cases as $L_{\infty\infty}$ or logics between $L_{\infty\omega}$ and $L_{\infty\kappa}$, see [3, 8, 13, 14].

Added in proof

The author has recently found a simpler proof of Corollary 5.7, without assuming that ω is the only measurable cardinal (see [24]).

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